



Universidad  
de Huelva

# On the Navier boundary condition for viscous fluids in a thin domain covered by very small asperities

F.J. Suárez-Grau

Departamento de Matemáticas, Universidad de Huelva, Spain

fjsgrau@us.es

## 1. Introduction

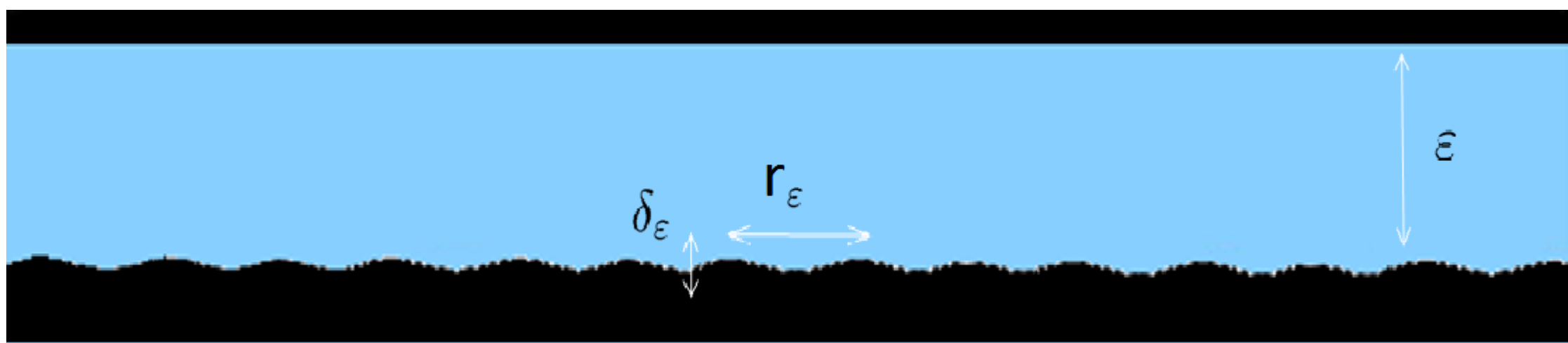
FOR a viscous fluid in an open set of  $\mathbb{R}^3$  with a rugous boundary, it is known that if the normal velocity vanishes on the boundary (Navier condition), then the fluid behaves as if the whole velocity vector vanishes on the boundary (adherence condition). This gives a mathematical explanation of why it is usual for a viscous fluid to impose the adherence condition. The equivalence between the Navier and adherence conditions was proved in [2] for a periodic rough boundary of small period  $\varepsilon$  and amplitude  $\varepsilon$ . In [3] it was considered the case of a weak roughness, namely the boundary was described by a periodic function of small period  $\varepsilon$  and amplitude  $\delta_\varepsilon$ , with  $\delta_\varepsilon/\varepsilon$  converging to zero.

Our aim in the present work is to study the relation between the Navier and adherence conditions in the case of a domain of small height  $\varepsilon$ . Namely, for a Lipschitz bounded open set  $\omega \subset \mathbb{R}^2$  and a function  $\Psi$  in  $W_{loc}^{2,\infty}(\mathbb{R}^2)$ , periodic of period  $Z' = (-1/2, 1/2)^2$ , we define  $\Omega_\varepsilon$  by

$$\Omega_\varepsilon = \left\{ x = (x', x_3) \in \omega \times \mathbb{R} : -\delta_\varepsilon \Psi\left(\frac{x'}{r_\varepsilon}\right) < x_3 < \varepsilon \right\}, \quad (1)$$

where the parameters  $r_\varepsilon, \delta_\varepsilon$  are chosen non-negative and satisfying

$$\lim_{\varepsilon \rightarrow 0} \frac{r_\varepsilon}{\varepsilon} = 0, \quad \lim_{\varepsilon \rightarrow 0} \frac{\delta_\varepsilon}{\varepsilon} = 0, \quad i.e. \quad \delta_\varepsilon \ll r_\varepsilon \ll \varepsilon.$$



We consider a fluid satisfying the Stokes system in  $\Omega_\varepsilon$ , the Navier condition on the rough boundary

$$\Gamma_\varepsilon = \left\{ x = (x', x_3) \in \omega \times \mathbb{R} : x_3 = -\delta_\varepsilon \Psi\left(\frac{x'}{r_\varepsilon}\right) \right\} \quad (2)$$

and (to simplify) the adherence condition on the rest of the boundary  $\partial\Omega_\varepsilon \setminus \Gamma_\varepsilon$ ,

$$\begin{cases} -\mu \Delta u_\varepsilon + \nabla p_\varepsilon = f & \text{in } \Omega_\varepsilon, & \text{div } u_\varepsilon = 0 & \text{in } \Omega_\varepsilon, \\ u_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon \setminus \Gamma_\varepsilon, \\ u_\varepsilon \cdot \nu = 0 & \text{on } \Gamma_\varepsilon, & \mu \frac{\partial u_\varepsilon}{\partial \nu} & \text{parallel to } \nu & \text{on } \Gamma_\varepsilon. \end{cases} \quad (3)$$

Here  $f = (f', f_3) \in L^2(\omega)^3$ ,  $\nu$  denotes the unitary outside normal vector to  $\Omega_\varepsilon$  in  $\Gamma_\varepsilon$  and  $\mu > 0$  corresponds to the viscosity of the fluid.

It is well known that (3) has a unique solution  $(u_\varepsilon, p_\varepsilon) \in H^1(\Omega_\varepsilon)^3 \times L_0^2(\Omega_\varepsilon)$  ( $L_0^2(\Omega_\varepsilon)$  denotes the space of functions in  $L^2(\Omega_\varepsilon)$  whose integral in  $\Omega_\varepsilon$  is zero). Moreover, we can show the following estimates

$$\int_{\Omega_\varepsilon} |u_\varepsilon|^2 dx \leq C\varepsilon^4, \quad \int_{\Omega_\varepsilon} |Du_\varepsilon|^2 dx \leq C\varepsilon^2, \quad \int_{\Omega_\varepsilon} |p_\varepsilon|^2 dx \leq C. \quad (4)$$

Our purpose is to study the asymptotic behavior of this system when  $\varepsilon$  tends to zero. We show that it depends on

$$\lambda = \lim_{\varepsilon \rightarrow 0} \frac{\delta_\varepsilon}{r_\varepsilon^{\frac{2}{3}}} \sqrt{\varepsilon} \in [0, +\infty]. \quad (5)$$

## 2. Changes of variables

OUR aim is to study the asymptotic behavior of  $u_\varepsilon$  and  $p_\varepsilon$  when  $\varepsilon$  tends to zero. For this purpose, we use a suitable combination of two changes of variables:

(1) Far of the rough boundary  $\Gamma_\varepsilon$  we use a dilatation in the variable  $x_3$  in order to have the functions defined in an open set of fixed height. Namely, we take  $\Omega = \omega \times (0, 1)$  and we define  $\tilde{u}_\varepsilon \in H^1(\Omega)^3$ ,  $\tilde{p}_\varepsilon \in L_0^2(\Omega)$  by

$$\tilde{u}_\varepsilon(y) = u_\varepsilon(y', \varepsilon y_3), \quad \tilde{p}_\varepsilon(y) = p_\varepsilon(y', \varepsilon y_3), \quad \text{a.e. } y \in \Omega. \quad (6)$$

(2) Near  $\Gamma_\varepsilon$  we use an original adaptation ([3]) of the *Unfolding Method* ([1], [5]), which is very related to the two-scale convergence method.



Near the rough boundary:  
Original adaptation of the Unfolding Method, see [3]

Far of the rough boundary:  
Dilatation defined by (6)

## 3. Main Result

Let  $(u_\varepsilon, p_\varepsilon) \in H^1(\Omega_\varepsilon)^3 \times L_0^2(\Omega_\varepsilon)$  be the solution of the Stokes system (3) and let  $\tilde{u}_\varepsilon, \tilde{p}_\varepsilon$  be defined by (6). Then, there exist  $v' \in H^1(0, 1; L^2(\omega))^2$ ,  $w \in H^2(0, 1; H^{-1}(\omega))$  and  $p \in L_0^2(\omega)$ , where  $p$  does not depend on  $y_3$ , such that, up to a subsequence,

$$\frac{\tilde{u}_\varepsilon}{\varepsilon^2} \rightharpoonup v' \text{ in } H^1(0, 1; L^2(\omega))^2, \quad \frac{\tilde{u}_{\varepsilon,3}}{\varepsilon^3} \rightharpoonup w \text{ in } H^2(0, 1; H^{-1}(\omega)),$$

$$\tilde{p}_\varepsilon \rightharpoonup p \text{ in } L^2(\Omega).$$

According to the value of  $\lambda$  defined by (5), we obtain the different expressions for  $v'$  and  $w$  depending on  $p$  which satisfies a *Reynolds equation*:

(i) If  $\lambda = +\infty$ , then denoting by  $P_{W^\perp}$  the orthogonal projection from  $\mathbb{R}^2$  to the orthogonal of the space  $W = \{\nabla_{z'} \Psi(z') \in \mathbb{R}^2 : z' \in Z'\}$ , we have that  $v'$  and  $p$  are given by

$$v'(y) = \frac{(y_3 - 1)}{2\mu} (y_3 I + P_{W^\perp}) (\nabla_{y'} p(y') - f'(y')), \quad \text{a.e. } y \in \Omega,$$

$$-\text{div}_{y'} \left( \left( \frac{1}{3} I + P_{W^\perp} \right) (\nabla_{y'} p - f') \right) = 0 \text{ in } \omega, \quad \left( \frac{1}{3} I + P_{W^\perp} \right) (\nabla_{y'} p - f') \cdot \nu = 0 \text{ on } \partial\omega.$$

$$\text{Moreover, the distribution } w \text{ is given by } w(y) = - \int_0^{y_3} \text{div}_{y'} v(y', s) ds, \text{ in } \Omega. \quad (7)$$

(ii) If  $\lambda \in (0, +\infty)$ , then defining  $(\hat{\phi}^i, \hat{q}^i)$ ,  $i = 1, 2$ , as solutions of the Stokes systems

$$\begin{cases} -\mu \Delta_z \hat{\phi}^i + \nabla_z \hat{q}^i = 0 & \text{in } \mathbb{R}^2 \times (0, +\infty), & \text{div}_z \hat{\phi}^i = 0 & \text{in } \mathbb{R}^2 \times (0, +\infty), \\ \hat{\phi}_3^i(z', 0) + \partial_{z_i} \Psi(z') = 0, & \partial_{z_3} (\hat{\phi}^i)'(z', 0) = 0, & \hat{\phi}^i(\cdot, z_3), & \hat{q}^i(\cdot, z_3) \text{ periodic of period } Z', \\ D_z \hat{\phi}^i \in L^2(Z' \times (0, +\infty))^{3 \times 3}, & \hat{q}^i \in L^2(Z' \times (0, +\infty)), \end{cases}$$

and  $R \in \mathbb{R}^{2 \times 2}$  by  $R_{ij} = \mu \int_{Z' \times (0, +\infty)} D_z \hat{\phi}^i : D_z \hat{\phi}^j dz$ ,  $\forall i, j \in \{1, 2\}$ , we have

$$v'(y) = \frac{(y_3 - 1)}{2\mu} \left( y_3 I + \left( I + \frac{\lambda^2}{\mu} R \right)^{-1} \right) (\nabla_{y'} p(y') - f'(y')), \quad \text{a.e. } y \in \Omega,$$

$$\text{where } p \text{ satisfies } \begin{cases} -\text{div}_{y'} \left( \left( \frac{1}{3} I + \left( I + \frac{\lambda^2}{\mu} R \right)^{-1} \right) (\nabla_{y'} p - f') \right) = 0 & \text{in } \omega, \\ \left( \frac{1}{3} I + \left( I + \frac{\lambda^2}{\mu} R \right)^{-1} \right) (\nabla_{y'} p - f') \cdot \nu = 0 & \text{on } \partial\omega. \end{cases}$$

Moreover, the distribution  $w$  is given by (7).

(iii) If  $\lambda = 0$ , then  $v'(y) = \frac{(y_3^2 - 1)}{2\mu} (\nabla_{y'} p(y') - f'(y'))$ ,  $\text{a.e. } y \in \Omega$ ,

$$\text{where } p \text{ satisfies } -\Delta_{y'} p = -\text{div}_{y'} f' \text{ in } \omega, \quad \frac{\partial p}{\partial \nu} = f' \cdot \nu \text{ on } \partial\omega.$$

Moreover, the distribution  $w$  is zero.

## 4. Conclusions

• For  $\lambda = +\infty$ , the main result shows that  $u_\varepsilon, p_\varepsilon$  behave as if in (3) we had assumed that  $\Gamma_\varepsilon$  was the plane boundary  $\{x_3 = 0\}$  and that the boundary condition on  $\Gamma_\varepsilon$  was

$$u_\varepsilon \in W^\perp \times \{0\} \text{ on } \Gamma_\varepsilon, \quad \partial_3 u'_\varepsilon \in W. \quad (8)$$

In particular, if  $W$  agrees with  $\mathbb{R}^2$  we deduce that the Navier condition in (3) is equivalent to the adherence condition  $u_\varepsilon = 0$  on  $\{x_3 = 0\}$ .

• For  $\lambda \in (0, +\infty)$ , the result shows that the asymptotic behavior of  $u_\varepsilon$  and  $p_\varepsilon$  is the same that if  $\Gamma_\varepsilon$  was the plane boundary  $\{x_3 = 0\}$  and the boundary condition on  $\Gamma_\varepsilon$  was

$$u_{\varepsilon,3} = 0 \text{ on } \Gamma_\varepsilon, \quad -\mu \partial_3 u'_\varepsilon + \lambda^2 R u'_\varepsilon = 0 \text{ on } \Gamma_\varepsilon, \quad (9)$$

i.e. although the roughness is not strong enough to deduce that the Navier condition on  $\Gamma_\varepsilon$  is equivalent to (8), it is sufficient to provide the friction coefficient  $\lambda^2 R u'_\varepsilon$  in (9).

• For  $\lambda = 0$ , the roughness is so weak that  $u_\varepsilon$  and  $p_\varepsilon$  behave as if  $\Gamma_\varepsilon$  was plane.

The critical size  $\lambda \in (0, +\infty)$  can be considered as the general one. In fact, the cases  $\lambda = 0$  and  $\lambda = +\infty$  can be obtained from this one by taking the limit when  $\lambda$  tends to zero and infinity respectively.

## References

- [1] T. Arbogast, J. Douglas, U. Hornung, *Derivation of the double porosity model of single phase flow via homogenization theory*, SIAM J. Math. Anal. 21 (1990) 823-836.
- [2] J. Casado-Díaz, E. Fernández-Cara, J. Simon, *Why viscous fluids adhere to rugose walls: A mathematical explanation*, J. Differential Equations 189 (2003) 526-537.
- [3] J. Casado-Díaz, M. Luna-Laynez, F.J. Suárez-Grau, *Asymptotic behavior of a viscous fluid with slip boundary conditions on a slightly rough wall*, M3AS 20 (2010) 121-156.
- [4] J. Casado-Díaz, M. Luna-Laynez, F.J. Suárez-Grau, *A viscous fluid in a thin domain satisfying the slip condition on a slightly rough boundary*, C. R. Acad. Sci. Paris, Ser. I (2010).
- [5] D. Cioranescu, A. Damlamian, G. Griso, *Periodic unfolding and homogenization*, C.R. Acad. Sci. Paris, Ser. I 335 (2002) 99-104.